Self-Alignment Mechanisms for Assistive Wearable Robots: a Kineto-Static Compatibility Method

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Abstract—The field of wearable robotics is gaining momentum thanks to its potential application in rehabilitation engineering, assistive robotics, and power augmentation. These devices are designed to be used in direct contact with the user, to aid the movement or increase the power of specific skeletal joints. The design of the so-called physical human robot interface is critical, since it determines not only the efficacy of the robot, but also the kinematic compatibility of the device with the human skeleton, and the degree of adaptation to different anthropometries. Failing to deal with these problems causes misalignments between the robot and the user joint. Axes misalignment leads to the impossibility of controlling the torque effectively transmitted to the user joint, and causes undesired loading forces on articulations and soft tissues. In this paper, we propose a general analytical method for the design of exoskeletons able to assist human joints without being subjected to misalignment effects. This method is based on a kineto-static analysis of a coupled mechanism (robot-human skeleton), and can be applied in the design of self-aligning mechanisms. The method is exemplified in the design of an assistive robotic chain for a two degrees-of-freedom human articulation.

Index Terms—Axes misalignment, exoskeleton, kinematics, mechanism design, rehabilitation robotics, wearable.

I. INTRODUCTION

POWERED exoskeletons are wearable robots designed to assist human movements either for augmenting the performance of healthy persons (e.g. endurance [1], [2] and strength enhancement [3]), restoring normal abilities in patients affected by movement disorders (e.g. tremor [4], hemiplegia [5], paraplegia [6]) or providing controlled rehabilitation therapy [7], [8]. Despite the different application fields, a common objective for all exoskeletons is to transfer a controlled amount of power to the user’s limb, monitoring its position at the same time. In most cases, a controlled physical interaction is needed at the joint level to finely support the user movement [9]. In rehabilitation, for example, the capability of controlling the trajectory or torque of each user’s joint independently is a great advantage to provide a more effective therapy [10]. In assistive applications, only the impaired joints should be artificially supported, keeping the others unaffected by the robot. Controlling the physical human-robot interaction at the joint level poses several problems in the mechanical design of exoskeletons, which arise mainly from the kinematical complexity and variability of the muscle-skeletal system. The design strategy for the user interface covers therefore a very important role in this issue, affecting both the efficiency and the safety of the exoskeleton [11].

The most critical issue in this process is the human-robot axes misalignment [12]. In order to transfer the desired torque to the user’s joint, the relative position of the instantaneous centre of rotation (ICR) of the human joint, with respect to the ICR of the robot-actuated joint, should be fully defined by the geometry and posture of the closed chain created by the connection of the robotic structure with the user’s limb. If this is not true, the ICR are misaligned and the transfer of torque is compromised. A correct alignment between the two axes is very difficult to obtain in practice, because of the nature of the human musculoskeletal system. Soft tissue (i.e., skin, muscle) deformations, inter- and intra-subject variability affect position and orientation of the axes of any human articulation, making almost impossible to build an accurate and reliable model of it. Human articulations are influenced by many factors that can change their geometrical description [13]. A wide inter-subject variability is present, due to differences in size and shape of bones. In addition, other intra-subject variability, depending on the looseness of the articulation, lead to a joint orientation which is not fixed, but depends on the loading and constraining condition. Moreover, the soft tissues are highly deformable, so that the robot attachment itself has a variable position with respect to the skeletal system. As an overall result, the relative position and orientation of the human joint axis with respect to the exoskeleton geometry can only be roughly estimated. If no compliance between the user and the exoskeleton is present, this error can lead to large “parasite” loads, which make the exoskeleton unusable and potentially dangerous for users.

The introduction of compliance in the connection between the exoskeleton mechanism and the user’s limb has been a common way of getting round the problem of axes misalignment in the past years [9]. This was achieved by imposing a single contact point between the robot and the user, usually localized at the end-effector (i.e., the hand [14], or the foot [15], [16]), or, in the case of multiple connection points between the limb and the exoskeleton, by using flexible attachments, such as cuffs [17], [18] or orthoses [19]. Commonly, exoskeletons feature manual regulations of link lengths, to fit the specific...
In this paper, we propose a complete analytical treatment of the force/torque transmission problem. In addition, it lacks a formally complete chain that cannot comply with the movement of the human ICR, which is a strong limitation, since an anthropomorphic robotic hand that replicates the body kinematics. As shown in [24], [26] is quite general, it cannot be applied when the robot body lead to a reduced mobility. While the approach in [21], [22], [23], [24], [25] includes the main DOFs of the joint, as well as additional DOFs to convey the effects of misalignments and variable or unknown quantities. The reference model is the white joint (\( \tilde{q} \)), while the additional joints representing misalignments are in grey (\( \delta \)). Reference points (\( O_F, O_E, E, E_O \)) of the system schematization are depicted, together with nominal lengths \( S_i \).

Fig. 1. Schematization of a wearable robot. (a) The robot designed to assist an articulation (e.g., elbow) is attached to the corresponding body segments (e.g., arm and forearm). Several variables (segment lengths, position/orientation of attachments, muscle and skin thickness) can affect the coupling. (b) The physical interface is schematized as a rigid connection between the robot extreme links and the human skeleton. The kinematic model of the articulation (c) includes the main DOFs of the joint, as well as additional DOFs to convey the effects of misalignments and variable or unknown quantities. The reference model is the white joint (\( \tilde{q} \)), while the additional joints representing misalignments are in grey (\( \delta \)). Reference points (\( O_F, O_E, E, E_O \)) of the system schematization are depicted, together with nominal lengths \( S_i \).

Proper solutions for the problem of axes misalignments have been proposed in several recent upper-limb exoskeletons [12], [21], [22], [23], [24], [25]. All these devices feature a one-to-one correspondence between human and robot joints, but introduce the use of supplementary passive degrees of freedom (DOFs) in the kinematic chain. Passive DOFs are introduced in the robot between two subsequent active joints, or to connect the robot links to the user limbs. This way, macroscopic misalignments between the robotic chain and the limb can be compensated. Although these works represent a clear advancement towards the solution of the misalignment problem, several limitations are still present. Although these works represent some advancements in the solution of the misalignment problem, they are focused on a specific case of study, rather than trying to find a general solution for the problem.

A general treatment of hyperstaticity in exoskeleton connections has been given more recently in [26]. This work showed how to add passive DOFs to the attachment points of robots to comply with hyperstaticity. When the exoskeleton structure replicates the limb kinematics, rigid connections with the body lead to a reduced mobility. While the approach in [26] is quite general, it cannot be applied when the robot does not replicate the body kinematics. As shown in [24], this is a strong limitation, since an anthropomorphic robotic chain cannot comply with the movement of the human ICR (neglected in [26]). In addition, it lacks a formally complete treatment of the force/torque transmission problem.

In this paper, we propose a complete analytical treatment of the problem of misalignments between a robotic linkage and an articulation to be assisted. A formal definition and approach to the following concepts is given: (1) kinematical adaptability (i.e., the capability of the exoskeleton to comply with axes misalignments); (2) observability of the human-robot dynamic system; (3) effectiveness and safety in providing torque to the user limb (i.e., to transfer the desired torque to the human joint without loading the articulation). The proposed design framework could be used as a general means for the solution of the axes misalignment problem, regardless of the specific application of the intended exoskeleton. A proof of concept of the proposed method is given in the paper through the design of a robotic linkage for the assistance of an articulation having two orthogonal revolute joints (2R) (e.g., the finger metacarpophalangeal joint). This paper is organized as follows: Section II presents a formalization of the misalignment problem. Section III carries on a general kineto-static analysis of the coupled chains. We provide some synthetic analytical steps for the analysis and the design of misalignments-free exoskeletons. To show the potential of this method, a case-study design for a 2R human articulation is presented in Section IV, together with a technique to split a three-dimensional (3D) analysis into simpler two-dimensional (2D) problems, analyzed in Section V. Finally, a discussion of the method and its potential applications are given in Section VI.

II. HUMAN-ROBOT JOINT ALIGNMENT

A. Introducing human and robotic chains

This Section aims to find a functional description for the alignment of two coupled kinematic chains. The first represents the human articulation. This chain has a certain mobility along its main degrees of freedom (DOFs), the so-called natural workspace, but is affected by uncertainties in its descriptive geometrical parameters (i.e., lengths, offsets, twists). These parameters reflect the variability of the human
anthropometry, the deformations of soft tissues and the presence of spurious motions of the articulation’s ICR around its reference position [22], [24]. This paper focuses on the case of two limbs, proximal and distal, connected by a single joint which provides multiple DOFs, i.e. a single articulation. The second chain models the robot, which should provide torques along the human DOFs, without being affected by variations of the human chain geometry. We will refer to such a robot as a self-alignment mechanism (SAM) [12], since our analysis deals with the mechanical description and axes alignment properties of the robotic chain. Our analysis will be focused only on serial SAMs, which are preferable in terms of controllability, modeling, and wearing comfort. The SAM is connected to the human chain at the proximal and distal segments, so that a closed kinematic structure is formed.

The human articulation is represented by a simplified kinematic model (the “reference model”), involving a set of DOFs of interest, together with nominal anthropometric sizes: typical models are the 1-DOF hinge joint (elbow or knee, [2], [3], [9], [12]), the 2-DOF gimbal joint (metacarpophalangeal articulations, [27], [28]), and the 3-DOF ball-socket joint (shoulder and hip, [9], [12], [14], [23]).

Such reference model is the starting point for the design of the exoskeletal chain. Misalignments between the SAM and the human chain occur when the human joint axes are in a different configuration than the one hypothesized by the reference model. This difference can always be interpreted as the effect of additional, unmodeled DOFs around the human articulation, which translate and rotate the human joint centre of unknown small quantities. This way of modelling misalignments is depicted in Fig. 1(c). It is worth to note that any condition leading to misalignments, such as the variability of human links lengths and the offsets and twists of the human joints axes, can be represented by means of these additional DOFs. Hence, the connections between SAM and the human chain will be treated as rigid (see Fig. 1).

B. Geometric description

Both chains will be described following the Denavit-Hartenberg notation. A tilde accent is used to indicate the human chain (H), and to distinguish it from the SAM1. As shown in Fig. 2, the two serial chains begin respectively from the points O0 and O0 laying on the proximal link, and end on the points E and E, laying on the distal link. The human joint reference model is described by a set of joint variables q and a set of given dimensions S, while the SAM joint variables are q. The aim of the SAM design is formally expressed by the following equivalence:

$$F T_e^{(SAM)}(q) = F T_e^{(H)}(q, S),$$

where $F$ refers to a fixed global reference frame ($O_F - x_F, y_F, z_F$). In order to comply with misalignments, this equation has to take into account possible deviations from the reference model, otherwise equality cannot hold. In the following, The additional joints, representing deviations from the reference model, will be addressed as “deviations” or “misaligning joints” and their variables will be indicated with $\delta$. The closure of the complete mechanism is then expressed as:

$$F T_e^{(SAM)} = F T_0(S) F T_e(q) = F T_e^{(H)} = F T_0(S, \delta) F T_e(q, \delta) F T_e(S).$$

- $F T_0(S)$ determines the relative positions of the attachment point of the robot on the proximal link.
- $T_e(q)$ is defined by the robot posture, and is therefore known except for the SAM mechanical imperfection (tolerances, backlashs, etc), which can be neglected.
- $T_0(S) T_e(q, \delta)$ defines the position of the $H$ chain initial frame.
- $T_e(S)$ expresses the connect between the exoskeleton and the human distal link.

C. Observations on variables definition

Equation (1) expresses the coincidence of ($E - x_e, y_e, z_e$) frame position and orientation as seen from each of the chains ($H$ and $SAM$). This is expressed by means of a minimal set of $N$ variables, with $N = 6$ in 3D space and $N = 3$ in 2D space. However, the totality of SAM’s DOFs ($\dim(q) = n$) may be greater than $N$, allowing the SAM chain to be redundant (i.e. $n > N$), if needed to satisfy other design constraints.

On the other hand, (1) can only be solved2 if a minimal equivalent set of variables is chosen for the human chain, $H$. This is shown for a 2D ($N = 3$) example in Fig. 3. While multiple sources of deviations $\delta$ exist (e.g. segment

1The human joints axis and the SAM one are denoted as $z_i$ and $\tilde{z}_i$, respectively.

2Matrix equation (1) is equivalent to $N$ independent scalar equations, due to inner constraints on the rotation matrix elements.
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The requirement should hold robustly for small variations of $\delta$, and for different sizes $S$. The closure constraint (1) can be also expressed as

$$0T_e(q) = (F T_0)^{-1} F T_0 \delta 0T_e(q) \delta T_e = 0T_e(S, \delta) \delta T_e(S) \Delta M(S, \delta, q) \quad (3)$$

where $0T_0$ collects the offsets due to sizes $S$ and the displacement $\delta$ of the human joint centre. Given the objective chain sizes $S$, $\text{SAM}$ should lead, via solution of (1), to a continuous functional relation between $q$ and $(\delta, q)$. Such function will be denoted as

$$(\delta, q) \Rightarrow f(q) \Leftrightarrow 0T_e(q) = M(S, \delta, q). \quad (4)$$

In order to guarantee continuity, $\text{SAM}$ must avoid singularities. It may happen that $f$ is not invertible, i.e., the $\text{SAM}$ structure is redundant ($n > N$). Redundancies may help the platform in satisfying external constraints or requirements [32].

### B. Evaluation of adaptability performance

Let us denote with $\Omega \subset \mathbb{R}^n$ the $\text{SAM}$ workspace, enclosing the $q$ which comply to the joint limitations (mechanical stops or singularities threshold)$^3$. Then the set

$$\Gamma = f(\Omega)$$

collects the reachable $(\delta, q)$ points. As stated before, our $\text{SAM}$ is designed for a reference geometry (correspondent to $\delta = 0$), but is required to be efficient for a range of geometry variations belonging to a neighbourhood $\mathcal{I}$ of 0, given by

$$\delta \in \mathcal{I} = [\delta(1,\min); \delta(1,\max)] \times [\delta(2,\min); \delta(1,\max)] \times \ldots \times [\delta(N-n,\min); \delta(N-n,\max)].$$

The set

$$\hat{\Omega}_\delta \overset{\Delta}{=} \{q : (\delta, q) \in \Gamma\} \subset \mathbb{R}^n$$

$^3\Omega$ is closed and simply connected, since it is given by a product of real intervals, $\Omega = [q_1,\min; q_1,\max] \times [q_2,\min; q_2,\max] \times \ldots \times [q_n,\min; q_n,\max]$. $\Gamma$ as well is closed and simply connected, since $f$ is continuous.
is a subset of \( \Gamma \) expressing the reachable human joint workspace, when the human articulation is in a particular configuration in the neighborhood of the reference one (that is, for a given \( \delta \)). The aim is to have a SAM structure such that

\[
\bigcap_{\delta \in \mathcal{I}} \hat{\Omega}_\delta \supseteq \hat{\Omega}_{\text{des}},
\]

where \( \hat{\Omega}_{\text{des}} \) is the set of human chain postures the SAM will be able to reach regardless of the human chain variations. Clearly, this subset of the natural workspace depends on the specific task for which the robot is designed. There are three elements in (5) that can be set by the designer: the desired workspace \( \hat{\Omega}_{\text{des}} \), the human articulation variability \( \mathcal{I} \) the SAM should comply with, and the SAM chain itself (hence the \( f \) function and the \( \Gamma \) set). If (5) does not hold, either a smaller \( \hat{\Omega}_{\text{des}} \) or a smaller articulation variability \( \mathcal{I} \) can be chosen (requirements relaxation). Alternatively, a different SAM structure can be used.

C. SAM joints partition

Besides satisfying the workspace requirements, the SAM should allow to build a correspondence between its actuated joint variables and the human chain posture \( \tilde{q} \), regardless of the misaligning \( \delta \) (inside the \( \mathcal{I} \) set). Starting from (4), function \( f \) is split into \( g \) and \( h \) functions, by separating the \( \tilde{q} \) and \( \delta \) expressions

\[
(\delta, \tilde{q}) = f(q) \Rightarrow \begin{cases} \tilde{q} = g(q) \\ \delta = h(q) \end{cases}.
\]

We will now draw a sufficient condition to separate the effects of \( q \) on \( \tilde{q} \) and on \( \delta \). We define a partition of the SAM variables \( q \in \Omega \subseteq \mathbb{R}^n \) by the following:

\[
q = \begin{bmatrix} q_a \newline q_c \end{bmatrix} : \text{dim}(q_c) = r \Rightarrow \text{dim}(q_a) = n - r,
q_a \in \Omega_a \subseteq \mathbb{R}^{n-r}, q_c \in \Omega_c \subseteq \mathbb{R}^r.
\]  

We will call \( q_c \), and the corresponding SAM joints, controlling variables (joints), and \( q_a \), the adaptive variables (joints). This partition yields to an equivalent division within the functions \( g \) and \( h \)

\[
\begin{cases} \hat{q} = g(q_a, q_c) \\ \hat{\delta} = h(q_a, q_c) \end{cases}.
\]

By properly choosing the partition (7), the human DOFs will be mainly dependent on the controlling variables \( q_c \). The concept of “proper” partition can be better expressed by analyzing the dependence of \( \tilde{q} \) on \( q_a \) and \( q_c \), by differentiation:

\[
\tilde{q} = g(q_a, q_c) \Rightarrow \hat{\tilde{q}} = \begin{bmatrix} G_0(q) \\ G(q) \end{bmatrix} \frac{\partial q_a}{\partial q_c} = G_0(q)q_a + G(q)q_c,
G_0 = \frac{\partial g}{\partial q_a} \in \mathbb{R}^{n-r \times n-r}, G = \frac{\partial g}{\partial q_c} \in \mathbb{R}^{n-r \times r}.
\]

By imposing that the contribution of the matrix \( G_0 \) is negligible (into the workspace \( \Omega \)) compared to the contribution of \( G \), we approximate the function \( g \) as a function of the sole variable \( q_c \). For the above matrices such condition can be expressed by means of the induced 2-norm (maximum singular value)

\[
\forall q \in \Omega, \|G_0(q)\|_2 \ll \|G(q)\|_2 \Rightarrow G_0(q) \simeq 0_{n \times (n-r)},
\]

leading (8) into

\[
\begin{cases} \hat{\tilde{q}} = g(q_c) \\ \hat{\delta} = h(q_a, q_c) \end{cases}.
\]

This formulation implies that the human joint posture \( q \) can be robustly identified through the SAM variables \( q_c \), regardless of the adaptive variables \( q_a \), assume in adjusting for the misalignment. It is worth to note that a full separation (i.e. \( \delta \) depending solely on \( q_a \)) is not necessary in order to achieve the human joints controllability.

Some considerations on the dimensions of partition (7) should be made:

- For \( \tilde{q} \) to be controllable, at least an equal number of controlling variables \( q_a \) should be provided:

\[
\text{dim}(q_c) = r \geq \hat{n} = \text{dim}(\hat{\tilde{q}}).
\]

- The number of adaptive variables \( q_a \) should be not-greater than the number of misaligning DOFs \( \delta \). Otherwise, when \( q_c \) (and then \( \tilde{q} \)) and \( \delta \) are given, some adaptive DOFs can change configuration, even if the human chain remains fixed (i.e. the SAM chain is underconstrained):

\[
\text{dim}(q_a) = n - r \leq N - \hat{n} = \text{dim}(\delta).
\]

D. Effectiveness of the assistance

The last requirement is more directly related to human assistance: SAM should be able to exert the desired assistive torques along the human joints \( \tilde{q} \), without inducing loads on the misaligning DOFs \( \delta \). Given the SAM joints partition of the previous section, if \( \tilde{q} \) is mainly defined by \( q_c \), as stated in (10), the desired torques at \( \tilde{q} \) are obtained by commanding proper torques at the SAM joints \( q_c \) (i.e. kineto-static duality). Calculations for the general case are not trivial, though, and are given in the upcoming section.

1) Kineto-static duality: As already done in (9), it is useful to separate adapting and controlling contribution in \( h \) in the second of (10):

\[
\delta = h(q_a, q_c) \Rightarrow \hat{\delta} = \begin{bmatrix} H_1(q) \\ H_2(q) \end{bmatrix} \frac{\partial q_a}{\partial q_c} = H_1(q)q_a + H_2(q)q_c,
H_1 = \frac{\partial h}{\partial q_a} \in \mathbb{R}^{(N-\hat{n}) \times (n-r)}, H_2 = \frac{\partial h}{\partial q_c} \in \mathbb{R}^{(N-\hat{n}) \times r}.
\]

Differentiation of (10) yields therefore to

\[
\begin{cases} \hat{\tilde{q}} = G(q_c)q_c \\ \hat{\delta} = H_1(q)q_a + H_2(q)q_c \end{cases}.
\]

Matrices \( G \), \( H_1 \) and \( H_2 \) can be considered the Jacobian matrices of the close kinematic. Matrices \( H_1 \) and \( G \) determine the SAM efficiency performances, and should satisfy the following conditions:
• $G$ must achieve maximum rank to ensure the independent control of each element of $\tilde{q}$. By incorporating (11) we obtain:

$$\text{rank}(G) = \tilde{n} \leq r. \quad (15)$$

• $H_1$ must achieve maximum rank, otherwise at least one non-trivial linear combination of $\tau_a$ is ineffective in the adaptation, corresponding to at least one useless DOF. An equivalent condition implied by (12) is

$$\text{rank}(H_1) = n - r \leq N - \tilde{n}. \quad (16)$$

A static dual expression of (14) can also be derived. Two geometric Jacobians can be built from (3), which express the same end-effector link velocity:

$$v_c = J\dot{\tilde{q}}, \quad \frac{\dot{\tilde{q}}}{\delta} = [ \tilde{J}_1 \tilde{J}_2 ] \begin{bmatrix} \dot{\tilde{q}} \\ \delta \end{bmatrix} = \begin{bmatrix} \tilde{j}_1 \tilde{j}_2 \end{bmatrix} \begin{bmatrix} \tilde{q}_a \\ \tilde{q}_c \end{bmatrix} = J_\delta = J(\tilde{q}) \dot{\tilde{q}}_c = J(\tilde{q}) \begin{bmatrix} q_a \\ \tilde{q}_c \end{bmatrix} \quad (17)$$

Following II-C, the human-chain transformation implies a total of $N$ variables, and $\tilde{J}$ is square ($N \times N$) and invertible. Its portions have dimensions $J_1 \in \mathbb{R}^{N \times \tilde{n}}$, $J_2 \in \mathbb{R}^{N \times (N - \tilde{n})}$. Substituting expressions (9) and (13) into (17) yields to

$$J\dot{q} = \tilde{J}_1\dot{\tilde{q}} + \tilde{J}_2\delta = \begin{bmatrix} \tilde{J}_1 & 0_\tilde{n} \times (n - r) \\ H_1 & H_2 \end{bmatrix} \begin{bmatrix} \dot{\tilde{q}} \\ \delta \end{bmatrix} \begin{bmatrix} \tilde{q}_a \\ \tilde{q}_c \end{bmatrix} = \begin{bmatrix} \tilde{J}_2H_1 & \tilde{J}_1G + \tilde{J}_2H_2 \end{bmatrix} \begin{bmatrix} \tilde{q}_a \\ \tilde{q}_c \end{bmatrix} \Rightarrow

J = \begin{bmatrix} \tilde{J}_2H_1 & \tilde{J}_1G + \tilde{J}_2H_2 \end{bmatrix} \in \mathbb{R}^{n \times (n - r)} \quad (18)$$

Given any end-effector wrench $f_c$, the dual expressions for (17) are

$$\tau = \begin{bmatrix} \tau_a \\ \tau_c \end{bmatrix} = J^Tf_c = J^T\tilde{\tau} = J^Tf_c = J^Tf_c \begin{bmatrix} \tau_a \\ \tau_c \end{bmatrix} \quad (19)$$

By inverting $\tilde{J}$, a relation between the SAM torques and the human torques can be obtained:

$$\begin{bmatrix} \tau_a \\ \tau_c \end{bmatrix} = J^Tf_c = J^T\tilde{J}^{-T} \begin{bmatrix} \tilde{\tau} \\ \tau_\delta \end{bmatrix} \Rightarrow

\begin{bmatrix} \tau_a \\ \tau_c \end{bmatrix} = H_1^TJ_2^T \tilde{J}^{-T} \begin{bmatrix} \tilde{\tau} \\ \tau_\delta \end{bmatrix} + H_2^TJ_2^T \tilde{J}^{-T} \begin{bmatrix} \tilde{\tau} \\ \tau_\delta \end{bmatrix} \quad (19)$$

To further simplify (19), it can be noticed that the products $J_1^T \tilde{J}^{-T}$ and $J_2^T \tilde{J}^{-T}$ are block matrices whose blocks are equal to either identity or empty matrices$^4$:

$$J^TJ^{-T} = I_{n_x} \begin{bmatrix} J_1^T \\ J_2^T \end{bmatrix} J^{-T} \begin{bmatrix} I_\tilde{n} \\ 0_{\tilde{n} \times (N - \tilde{n})} \end{bmatrix} \begin{bmatrix} J_1 \tilde{J}_2 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} \tau_a \\ \tau_c \end{bmatrix} = H_1^TJ_2^T \tilde{J}^{-T} \begin{bmatrix} \tilde{\tau} \\ \tau_\delta \end{bmatrix} = G^T \tilde{\tau} + H_2^T \tilde{J}_2^T \tilde{J}^{-T} \begin{bmatrix} \tilde{\tau} \\ \tau_\delta \end{bmatrix} \quad (20)$$

Equation (20) is the static dual of the kinematic (14). It states that a desired assistance on the main human torques $\tilde{\tau}$ affects only the control joints torques $\tau_c$ and not the adaptive ones $\tau_a$, and that having no torques acting on the misaligning DOFs implies that the adaptive joints are passive: $(\tau_\delta = 0 \Rightarrow \tau_a = 0)$. The opposite is not straightforward: by leaving the adaptive SAM joints passive $(\tau_a = 0)$ and commanding the proper $\tau_c$ may not always correspond to having the desired $\tilde{\tau}$ while relieving the misaligning DOFs. Partition between active and passive joints$^{[33],[34]}$ should be then carefully evaluated. A particular case arises when condition (16) holds equal. In this case matrix $H_1$ is full-rank square (thus invertible), and adaptive joints are as many as the misaligning DOFs. This is the case in which$^{[26]}$ falls, but only if, adopting their robotic joints separation, the equation (8) can be well approximated by (10). The first row of (20) then ensures the equivalency between having unloaded deviation $\delta$ on human and passive adaptive DOFs on SAM:

$$\text{rank}(H_1) = n - \tilde{n} = n - r \neq 0 \Rightarrow \tau_\delta = 0 \Leftrightarrow \tau_a = 0. \quad (21)$$

2) Assistance torques: The formalizations that we have introduced cover a more general case: if $H_1$ is not square, the correspondence between $\delta$ torques and adaptive torques is only partial and not invertible. This section will address an actuation strategy allowing the SAM to relate given torques on the controlling joints with the desired torques on the human joint, without loading the misaligning DOFs. Condition (16) implies that, under proper columns rearrangements$^5$, $H_1^T \in \mathbb{R}^{(n-r) \times (N-\tilde{n})}$ can be divided into two aligned blocks, a square full-rank (and so invertible) block and a rectangular block being a linear combination of the square one:

$$H_1^T = \begin{bmatrix} H_1^T \quad H_1^T \end{bmatrix} \text{ with } H_1^T \in \mathbb{R}^{(n-r) \times (N-\tilde{n})}, \quad \exists A \in \mathbb{R}^{(n-r) \times (N-\tilde{n})} : H_1^TA = 0_{(n-r) \times (N-\tilde{n})}$$

The first of (20) can be rewritten, partitioning $\tau_\delta$ into two subsets $\tau_{\delta1} \in \mathbb{R}^{N-\tilde{n}}$ and $\tau_{\delta2} \in \mathbb{R}^{N-\tilde{n}-(n-r)}$ compatible with the dimensions of the $H_1^T$ blocks, as

$$\tau_a = H_1^T \tau = \begin{bmatrix} H_1^T \\ -H_1^T \end{bmatrix} A \begin{bmatrix} \tau_{\delta1} \\ \tau_{\delta2} \end{bmatrix} = H_1^T \begin{bmatrix} I_{n-r} \\ -A \end{bmatrix} \begin{bmatrix} \tau_{\delta1} \\ \tau_{\delta2} \end{bmatrix}$$

Since $H_1^T$ is invertible, $\tau_a = 0$ condition corresponds to an inner linear dependence between the elements of $\tau_\delta$:

$$\tau_a = 0 \Rightarrow \tau_{\delta1} = A \tau_{\delta2} \quad (21)$$

This demonstrates that if the adaptive DOFs are less than the misaligning DOFs, a number $(N - \tilde{n}) - (n - r)$ of $\tau_\delta$ elements (in our notation they are $\tau_{\delta2}$) remains undetermined, and can only derive from the controlling joint torques $\tau_c$.}$^6$Such rearrangement is equivalent to change the order of the elements in $\tau_\delta$.

---

$^4$I_k indicate the $k \times k$ identity matrix, while $0_{l \times m}$ an empty matrix with $l$ rows and $m$ columns.

$^5$Such rearrangement is equivalent to change the order of the elements in $\tau_\delta$. 

$^6$
Conversely, since the controllable torques are more than the human joints (see (11)), we can find an actuation strategy such that \( r \) combinations of \( \tau_r \) are independent and work on \( \tilde{\tau} \), while the remaining \( r - \tilde{n} \) work on the \( \tau_{\delta 2} \) variables. In the following paragraph we show how such partition can be derived and the condition on which it is effective.

As done before for \( H^T_{11} \), we can divide \( G \) from equation (15) as follows:

\[
G = \begin{bmatrix} G' & G'' \end{bmatrix} \text{ with } G' \in \mathbb{R}^{\tilde{n} \times \tilde{r}}, \quad \exists B \in \mathbb{R}^{\tilde{n} \times (r - \tilde{n})} : G'' + G'B = 0_{\tilde{n} \times (r - \tilde{n})}.
\]

\( B \) can be used to build a combination block matrix

\[
T \triangleq \begin{bmatrix} I_{\tilde{n}} & 0_{\tilde{n} \times (r - \tilde{n})} \end{bmatrix} \in \mathbb{R}^{\tilde{n} \times r},
\]

and \( \tau_c \) can be divided accordingly to such block dimensions into \( \tau_{c1} \in \mathbb{R}^{\tilde{n}} \) and \( \tau_{c2} \in \mathbb{R}^{r - \tilde{n}} \). Multiplication of the second of (20) with \( T \) and substitution of (21) leads to

\[
T \tau_c = \begin{bmatrix} I_{\tilde{n}} & 0_{\tilde{n} \times (r - \tilde{n})} \end{bmatrix} \begin{bmatrix} \tau_{c1} \\ \tau_{c2} \end{bmatrix} = \begin{bmatrix} \tau_{c1} \\ B^T \tau_{c1} + \tau_{c2} \end{bmatrix} = TG^T \tilde{\tau} + TH^T_{2} \begin{bmatrix} \tau_{\delta 1} \\ \tau_{\delta 2} \end{bmatrix} = T \tau \delta 2.
\]

The product \( TG^T \) becomes

\[
TG^T = \begin{bmatrix} I_{\tilde{n}} & 0_{\tilde{n} \times (r - \tilde{n})} \end{bmatrix} \begin{bmatrix} G'^T \\ G''^T \end{bmatrix} = \begin{bmatrix} G'^T \\ B^T G^T + G''^T \end{bmatrix} = \begin{bmatrix} G'^T \\ 0_{(r - \tilde{n}) \times \tilde{n}} \end{bmatrix} \Rightarrow
\]

\[
\begin{aligned}
\tau_{c1} &= G'^T \tilde{\tau} + \begin{bmatrix} I_{\tilde{n}} & 0_{\tilde{n} \times (r - \tilde{n})} \end{bmatrix} H^T_{2} \begin{bmatrix} A \\ I_{N - \tilde{n} - (n - \tilde{n})} \end{bmatrix} \tau_{\delta 2} \\
B^T \tau_{c1} + \tau_{c2} &= \begin{bmatrix} B^T & I_{n - \tilde{n}} \\ I_{N - \tilde{n} - (n - \tilde{n})} & I_{N - \tilde{n} - (n - \tilde{n})} \end{bmatrix} \begin{bmatrix} A \\ I_{N - \tilde{n} - (n - \tilde{n})} \end{bmatrix} \tau_{\delta 2}.
\end{aligned}
\] (22)

Equations (22) show that if the active joints torques are divided between two subsets \( \tau_{c1} \) and \( \tau_{c2} \), the linear combination \( B^T \tau_{c1} + \tau_{c2} \triangleq \tau^*_c \) defines solely \( \tau_{\delta 2} \):

\[
\tau^*_c = \begin{bmatrix} B^T & I_{n - \tilde{n}} \end{bmatrix} H^T_{2} \begin{bmatrix} A \\ I_{N - \tilde{n} - (n - \tilde{n})} \end{bmatrix} \tau_{\delta 2} \triangleq X \tau_{\delta 2}
\]

The matrix \( X \in \mathbb{R}^{(r - \tilde{n}) \times (N - \tilde{n} - (n - \tilde{n}))} \) has at least as many rows as columns, since \( r - \tilde{n} \geq N - \tilde{n} - (n - \tilde{n}) \iff n \geq N \). So, if \( X \) achieves full rank,

\[
\text{rank}(X) = (N - \tilde{n}) - (n - \tilde{n}) \quad (23)
\]

it’s left-invertible, and therefore

\[
\tau^*_c = 0 \Rightarrow \tau_{\delta 2} = 0,
\]

which then implies from (21) and the first of (22),

\[
\tau_{\delta 1} = A \tau_{\delta 2} = 0, \quad \tau_{c1} = G'^T \tilde{\tau} \quad (24)
\]

As anticipated, even if the \( r \) active \( \text{SAM} \) joints are more than the \( \tilde{n} \) human joint DOFs, only a part of \( \tilde{n} \) torques is sufficient for the actuation (first equation of (22), last of (24)), while the remaining \( r - \tilde{n} \) are defined from \( \tau^*_c = 0 \Rightarrow \tau_{\delta 2} = -B^T \tau_{c1} \).

This is a case of underactuated system [35], [36]. The key point of the above discussion is that, assuming all matrices to have maximum rank, while the first of (20) cannot be always “inverted” because \( H^T_{11} \) may have not independent columns (i.e., \#columns \( \geq \#rows \)), this can be done with the second of (22) (for \( X, \#columns \leq \#rows \)). The rank of \( X \) tells us how many conditions can be imposed between the controlling torques in surplus from the required \( \tilde{\tau} \), and the misaligning DOFs torques remaining undetermined from \( \tau_a \).

E. Summary of requirements verification

This section summarizes the main requirements to be satisfied by a general wearable robot in order to assist a human joint that is subjected to variable misalignments conditions.

1) Choose a \( \text{SAM} \) structure which has at least \( n \geq N \) DOFs, where \( N \) is the operational workspace dimension.
2) Solve the kinematic closure: the function \( f \) in (4) is identified.
3) Evaluate the reachable workspace \( \Gamma = f(\Omega) \). Verify if its subset related to human joint variables satisfies equation (5).
4) By differentiation of \( f \), verify if human joint posture can be derived by a proper subset of \( \text{SAM} \) joints variables as expressed in equation (10). Such partition of \( \dot{q} \) must verify dimensional requirements (11) and (12).
5) Differentiation of relation (10) leads to (14): ranking condition (15) and (16) must be verified on the obtained Jacobians.
6) Equations (20) expresses the relation between human chain torques and \( \text{SAM} \) torques: the assistance requirement is to obtain desired torques \( \tilde{\tau} \) on the human joint DOFs, while not loading the misaligning DOFs, \( \tau_\delta = 0 \).
7) If \( \text{SAM} \) is “square” (i.e., \( r = \tilde{n} \) and \( N = n - \tilde{n} \)), matrices \( H^T \) and \( G^T \) are square, so equations (20) are sufficient to invert the torques relation by matrix inversion.
8) If \( \text{SAM} \) has arbitrary dimensions, matrices \( A \) and \( B \) shall be partitioned and the additional ranking condition (23) must be verified. If so, the assistive torques strategy is:

- Passive adaptive DOFs: \( \tau_a = 0 \Rightarrow \tau_{\delta 1} = A \tau_{\delta 2} \)

- Actuation strategy: \( \tau^*_c = B^T \tau_{c1} + \tau_{c2} = 0 \)

- Relieved misaligning DOFs: \( \tau_\delta = 0 \)

- Underactuation requirements: \( \tau_{c2} = -B^T \tau_{c1} \)

- Assistance-desired torques relation: \( \tau_{c1} = G'^T \tilde{\tau} \) (25)

IV. \text{SAM} DESIGN EXAMPLE: 2R HUMAN SPATIAL JOINT

Section III provided several requirements that a \( \text{SAM} \) chain should obey. However, the choice of a suitable kinematic chain (see Point 1 of Section “III-E. Summary of requirements verification”) is out of the scope of that framework, and is extremely difficult to generalize. In a typical design problem, several \textit{a priori} constraints are imposed before the choice of the kinematics of the robot; for example constraints on weight distribution, limitations introduced by the material, accessibility, interchangeability and aesthetics.
is used here for the mechanism design of a wearable robot. Specifically, we show the design of a SAM for a 2R DOFs is formally equivalent to the splitting of the spatial mechanism into two planar sub-mechanisms: hence we will refer to this approach as planar decoupling.

The initial SAM has $N = 6$, $\bar{n} = 2$, while the resulting two SAMs are planar and assist one joint at a time ($N = 3$, $\bar{n} = 1$). Consequently, the misaligning variables shall be redefined accordingly to the planar decoupling.

### A. SAM for 2-revolute DOFs human chain

Following the definitions introduced previously, $N = 6$ while the human joint variables are two, $\tilde{q} = [\tilde{q}_1, \tilde{q}_2]^T \in \mathbb{R}^2$. Then $\delta = [\delta_1, \delta_2, \varepsilon]^T \in \mathbb{R}^4$ are the misaligning variables, accounting for 3 translational terms and for the third rotation angle $\varepsilon$.

Referring to Fig. 4(a), the fixed frame $(\tilde{O}_0 - \tilde{x}_0, \tilde{y}_0, \tilde{z}_0)$ is centered in the 2-DOF joint centre, such that the axis $\tilde{z}_0$ lies along the first human joint axis, and the axis $\tilde{y}_0$ along the second one when $\tilde{q}_1$ is null. The axis $\tilde{x}_0$ closes the right-handed frame. The rotations $\tilde{q}_1$ and $\tilde{q}_2$, around the axis $\tilde{z}_0$ and $\tilde{z}_1$ respectively, move the original frame into the fixed reference frame of the human link, whose axes directions are so defined: $\tilde{x}_e$ lies parallel to the new direction of $\tilde{x}_0$; $\tilde{z}_e$ is chosen parallel to the joint axis $\tilde{z}_1$; $\tilde{y}_e$ closes the right-handed frame. The $\varepsilon$ contribution can be treated as the last rotation occurring along $\tilde{x}_e$.

By proper calculations, we can find the following formulation for the right-hand term in (3):

$$M(S, \delta, \tilde{q}) = \hat{0}T_{\bar{q}}(S, \delta)\hat{0}T_{e}(\tilde{q}, \delta)^T T_s(S) = \begin{bmatrix} R^0_\delta & \hat{l}_X(S) & \hat{l}_Y(S) & \hat{l}_Z(S) \\ l_\varepsilon & 1 \\ 0^T \\ T_0_e(\tilde{q}, 0) \end{bmatrix} \hat{0}T_{e}(\tilde{q}, 0) \begin{bmatrix} R^0_e \hat{l}_H(S) \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

(26)

where we choose to express $\delta$ along the fixed frames axes (the prismatic grey joints in Fig. 4(a)), and $l_X$, $l_Y$, and $l_Z$ are human-size related offsets, while $l_H$ is the offset between the $E$ attachment point and $\tilde{O}_0$ ($\tilde{0}$ represents the vector $[0, 0, 0]^T$).

More details can be found in Appendix A.

### B. Planar decoupling and SAM design

In order to get a satisfactory expression for the SAM decoupling, the $\tilde{q}$ components must be separated: referring to Fig. 4(b), planar decoupling lead us to work separately in the planes $\pi_1$ and $\pi_0$. Equation (26) lumps the $\varepsilon$ effects in a separated factor; if we temporarily ignore it, only two linear combination of the three displacements $\delta_i$ still matter when working in each of the planes.

In Fig. 4(b), a visual representation of an equivalent decomposition of $\delta$ in two vectors $\delta^*$ and $\delta'$ laying on the $\pi_1$, $\pi_0$ planes is given. Details of related calculations are given in Appendix A.
Appendix A. The main result is

$$
\begin{bmatrix}
    L_3 & \delta^1 \\
    0^T & 1
\end{bmatrix} T_{\text{eq}}(q, 0) =
\begin{bmatrix}
    \tilde{c}_1 & -\tilde{s}_1 & 0 & \delta_1^1 \\
    \tilde{s}_1 & \tilde{c}_1 & 0 & \delta_1^2 \\
    0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
    R_x(-\frac{\pi}{2}) & 0 \\
    0^T & 1
\end{bmatrix}
\begin{bmatrix}
    \tilde{c}_2 & -\tilde{s}_2 & 0 & \delta_2^1 \\
    \tilde{s}_2 & \tilde{c}_2 & 0 & \delta_2^2 \\
    0 & 0 & 1 & 0
\end{bmatrix}.
$$

The first and last term of (27) represent the planar transformations that rule the design of the reduced planar SAMs, while the middle term is an axes rearrangement. The last factor is the only term that matters in the plane \( \pi_1 \), while the first one affects the \( \pi_0 \) in-plane motions. However we observe that \( \delta_1^1 \) and \( \delta_1^2 \) depend on \( \tilde{q}_1 \). Considering the requirements in “III-A. Solution of kinematic closure” and “III-B. Evaluation of adaptability performance”, this dependency will only results in a modified neighborhood \( \tilde{I} \). For example, a condition for a suitable \( \tilde{I} \) could be \( |\delta| \leq \epsilon \), i.e. a maximum joint centre displacement \( \epsilon \) is assumed. Due to rotation properties, the \( \tilde{I} \) boundary remains the same, \( |\delta| = \epsilon \Leftrightarrow |\delta^T| = \epsilon \) whichever the \( \tilde{q}_1 \). The planar decoupling splits this condition in two sufficient but not necessary conditions:

- \( \pi_1 \)-coplanar SAM and \( |\delta^T| = \epsilon \) for the \( \pi_0 \)-coplanar one (see Fig. 4(b)). The neighbourhood \( \tilde{I} \) resulting from \( \left( \left| \begin{bmatrix} \delta_1^1 & \delta_2^1 \\ \delta_1^2 & \delta_2^2 \end{bmatrix} \right| \leq \epsilon \right) \) is larger than the starting one, so SAM is requested to show more adaptability performance than those effectively needed.

Despite that the planar decoupling approach simplifies the design process.

V. SAM DESIGN EXAMPLE: 1R HUMAN PLANAR JOINT

In this section we apply our methodology to a specific SAM structure. Specifically, we present a simple planar case, with a single revolute (human) joint, which can undergo displacements in the plane perpendicular to its axis. Clearly, this is not a complete example of SAM design for 1R joint. In fact, it neglects the out-of-plane misaligning displacement and the other two rotations, other than over-simplifying the human joint with a perfectly planar model. Nevertheless, it can be used in continuation with the planar decoupling approach to executing the design of a complete SAM structure: the planar decoupling approach simplifies a multi-dimensional problem in several independent planar problems, which are analysed in details in this section.

The SAM kinematic chain for this example is shown in Fig. 5, together with its reference frames: its structure is PRR, while the fourth joint \( R_3 \) represents the human joint. It is out of the scope of this paper to discuss about its advantages and disadvantages against other possible structures.

A. Kinematic compatibility

With reference to Fig. 5, \( H \) and \( l_h \) are body size related quantities (components of vector \( S \)), while \( l_0 \) is an offset for the position of the prismatic joint. Let us denote the desired workspace with \( \tilde{q} \in \tilde{\Omega}_{\text{des}} = [0, \frac{\pi}{2}] \). It can be noted that \( h > H > 0 \) and \( l_c > H > 0 \), in order to avoid interference with the human body, and that \( q_2 < \frac{\pi}{2} \), otherwise singularity occurs and it is not possible to reach \( \tilde{q} = \frac{\pi}{2} \). This last condition implies that \( l_b > h + l_h \) which is the maximum vertical distance between \( P_1 \) and \( E \).

Equation (3) can be solved in order to get the \( f \) expressions (details are in Appendix B):

\[
(q_1, q_2, q_3) \leftrightarrow \left\{ \begin{array}{l}
\delta_1 = (q_1 - l_0) + l_a + l_b c_2 + l_c e_{23} - l_h s_{23} \\
\delta_2 = l_b s_2 + l_c e_{23} - h + l_h c_{23}
\end{array} \right.
\]

Since the first of (28) does not depend on \( \delta_1 \), imposing (5) over (28) leads to the same conclusions whichever \( \tilde{I} \) is chosen: \( \Omega \) must include the subset \( q_2 + q_3 \in [\frac{\pi}{2}, \pi] \). Then, a limit to the \( \tilde{I} \) contour comes from degeneration of the other (28) relations. For the \( \delta_2 \) limit, for example, we will get

\[
\delta_2 = l_b (\sin q_2 - \sin q_{2,\text{REF}}) \leftrightarrow \delta_2 \leq \frac{l_b}{\sin q_{2,\text{REF}}} = \sin q_2 - \sin q_{2,\text{REF}}.
\]

where

\[
\sin q_{2,\text{REF}} = \frac{h - l_c \cos \tilde{q} + l_b \sin \tilde{q}}{l_b}.
\]

Limitations over \( \delta_2/l_b \) are then given by extreme values of \( \sin q_2 - \sin q_{2,\text{REF}} \):

\[
-1 - \sin q_{2,\text{REF}} < \frac{\delta_2}{l_b} < -1 - \sin q_{2,\text{REF}}.
\]

Choosing \( l_h \geq 0 \), the expression of \( \sin q_{2,\text{REF}} \) is monotonic in \( \tilde{q} \) between the values

\[
\frac{h - l_c}{l_b} \leq \sin q_{2,\text{REF}} \leq \frac{h + l_h}{l_b},
\]

which are contained in the \([-1; 1]\) set if

\[
l_h > \max \{(l_c - h, h + l_h)\}.
\]

Then, limitations above \( \delta_2/l_b \) are given by:

\[
\frac{\delta_{2,\max}}{l_b} = 1 - \min \{\sin q_{2,\text{REF}}\} = 1 - \frac{h - l_c}{l_b};
\]

\[
\frac{\delta_{2,\min}}{l_b} = -1 - \max \{\sin q_{2,\text{REF}}\} = -1 - \frac{h - l_h}{l_b}.
\]
Regarding the $\delta_1$, it can be easily allowed whatever its maximum value by enlargement of the prismatic joint stroke. In Fig. 6 the singularity condition related to $\delta_{2,max}$ is represented.

### B. Actuation effectiveness

Given that workspace requirements have been met, the assistance effectiveness shall be addressed. The division between $q_c$ and $q_a$ results straightforward from (28), with $r = 2$, $n - r = 1$: the dimensions satisfy (11) and (12). Functions $g$ and $h$ of (10) are obtained

\[
\begin{align*}
\mathbf{q} &= \left[ q_1 \right], \quad \mathbf{\delta} = \left[ \delta_1 \right], \quad \mathbf{q}_c = \left[ q_2 \quad q_3 \right], \quad \mathbf{q}_a = \left[ q_1 \right] \\
\mathbf{\hat{q}} &= g(\mathbf{q}_c) \leftrightarrow \mathbf{q} = q_2 + q_3 - \frac{\pi}{2} \\
\mathbf{\delta} &= \mathbf{h}(\mathbf{q}_c, \mathbf{q}_c) \leftrightarrow \begin{cases} 
\delta_1 = (q_1 - l_0) + l_a + l_b c_2 + \ldots \\
\delta_2 = l_b s_2 + l_c s_23 - h + l_b c_23
\end{cases}
\end{align*}
\]

Matrices $G$, $H_1$ and $H_2$ of (14) are

\[
\begin{align*}
G &= \frac{\partial q}{\partial q_c} = \begin{bmatrix} 1 & 1 \end{bmatrix}, \\
H_1 &= \frac{\partial h}{\partial q_a} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \\
H_2 &= \frac{\partial h}{\partial q_c} = \begin{bmatrix}
-l_b s_2 - l_c s_23 - l_b c_23 - l_c s_23 - l_b c_23 \\
l_b c_2 + l_c c_23 - l_b s_23 - l_c c_23 - l_b s_23
\end{bmatrix}.
\end{align*}
\]

Ranking conditions (15) and (16) give rank ($G$) = 1 = $\hat{n}$, rank ($H_1$) = 1 = $n - r$. This SAM is not “square”, so partitions for $G$ and $H_1^T$ must be expressed,

\[
H_1^T = \begin{bmatrix} 1 & 0 \end{bmatrix} \Rightarrow H_1^{rT} = [1], H_1^{r'\tau} = [0], A = [0]
\]

and

\[
G = \begin{bmatrix} 1 & 1 \end{bmatrix} \Rightarrow G' = [1], G'' = [1], B = [-1],
\]

and the arguments of (25) are recalled:

- partition of misaligning torques comes from $H_1^T$ blocks decomposition,

\[
\begin{align*}
\mathbf{\tau}_{\delta 1} &= \left[ \tau_{\delta 1} \right], \quad \mathbf{\tau}_{\delta 2} = \left[ \tau_{\delta 2} \right],
\end{align*}
\]

and leads to

\[
\begin{align*}
\mathbf{\tau}_a &= \left[ \tau_1 \right] = 0 \Rightarrow \mathbf{\tau}_{\delta 1} = \mathbf{A}\mathbf{\tau}_{\delta 2} = 0; \\
\text{• partition matrix for active torques is}
\end{align*}
\]

\[
T = \begin{bmatrix}
I_{\hat{n}} & 0_{\hat{n} \times (r-\hat{n})} \\
B^T & I_{r-\hat{n}}
\end{bmatrix} = \begin{bmatrix} 1 & 0 \\
-1 & 1 \end{bmatrix},
\]

and leads to

\[
\begin{align*}
\mathbf{\tau}_{c_1} &= \left[ \tau_{c_1} \right], \quad \mathbf{\tau}_{c_2} = \left[ \tau_{c_2} \right], \quad \mathbf{\tau}_{c}^* = \left[ -\tau_2 + \tau_3 \right]; \\
\text{• matrix } X \text{ results}
\end{align*}
\]

\[
X = \begin{bmatrix} B^T & I_{r-\hat{n}} \end{bmatrix} H_2^T \begin{bmatrix} A \\
I_{N-\hat{n}} \end{bmatrix} = \begin{bmatrix} \tau_{\delta 2} = 0 \Rightarrow \tau_{\delta 2} = 0 \Rightarrow \\
\tau_{c_1} = [\tau_2] = G^T \tau = [1] \tau = [\tau] \Rightarrow \\
\tau_2 = \tau \quad \tau_3 = \tau_2 = \tau \end{bmatrix}.
\]

Condition (23) becomes rank ($X$) = 1, which is equivalent to $c_2 < \frac{\pi}{2}$. We have therefore

\[
\begin{align*}
\begin{cases}
\mathbf{\tau}_c &= \left[ -\tau_2 + \tau_3 \right] = 0 \Rightarrow \mathbf{\tau}_{\delta 2} = \left[ \tau_{\delta 2} = 0 \Rightarrow \\
\mathbf{\tau}_{c_1} &= \left[ \tau_2 \right] = G^T \mathbf{\tau} = [1] \mathbf{\tau} = [\mathbf{\tau}] \Rightarrow
\end{cases}
\end{align*}
\]

Relations (30) suggests the actuation strategy that defines the required SAM controlling torques $\tau_2$ and $\tau_3$ when a desired torque along the human joint $\hat{\tau}$ is required. Equations (29) and (30) help in estimating the SAM performances: within a design process, they should be used for numerical evaluation of workspace, encumbrance, weight and actuation costs. Depending on the results the designers can choose if proceed with the actual architecture or try another different solution than the one presented in Fig. 5.

### C. Back to the 2R spatial joint

To complete the description of the planar decoupling approach, here we show how the proposed planar SAM structure can be exploited to complete the 3-D spatial mechanism. With reference to Fig. 4(b), we can design the reduced $\pi_0$ and $\pi_1$ SAMs independently: they both will build a 1-DOF single closed loop planar chain when linked with the respective $R$ human joint. The two-dimensional Kutzbach criterion [37] suggests that 4 links (including the fixed one) connected by four 1-DOF joints are necessary to achieve such solution:

\[
M = 3(n - 1 - j) + \sum_{i=1}^{j} f_i = 3(4 - 1 - 4) + 4 = 1.
\]

This will result in 4 joints for the $\pi_1$ section (including the $\hat{z}_1$ joint) and 4 joints for the $\pi_0$ section (including the $\hat{z}_0$ joint), for a total of 6 joints in the SAM chain: the three-dimensional Kutzbach criterion will give (7 joints, 6 with 1 DOF and the human one with 2 DOF, 7 link including the fixed one)

\[
M = 6(n - 1 - j) + \sum_{i=1}^{j} f_i = 6(7 - 1 - 7) + 8 = 2.
\]

Nevertheless, if the SAM structure is built in this way, it may be underconstrained. In fact, let us assume $\mathbf{q}$ and $\mathbf{\delta}$ fixed: the two points $O_0$ and $E$ are fixed. Referring to Fig. 7(a), the two planar sub-chains can be intended as having their end-effectors in $A_0$ and $A_1$ points. Each chain regulates its end-effector $x$, $y$ and orientation on the respective plane: the structure depicted in Fig. 7(a) is undetermined with respect to translation of the $\overline{A_0A_1}$ link along a direction parallel to both $\pi_0$ and $\pi_1$, i.e.
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P solution is to exploit the unconstrained direction just described, Therefore, one SAM dimension equal to 5, not 6, and the resultant mobility is 3.

Such joint works in both the planar and the (as in Fig. 2) planar prismatic joint is exploited even in π1-coplanar part. This solution is presented in Fig. 7(b), where the π1-coplanar mechanism is the same depicted in Fig. 5.

VI. DISCUSSION AND CONCLUSIONS

A. Discussion

The variability in the user anthropometry is the primary source of misalignments between the actuated axes of an exoskeletal robot and the wearer’s joint to be assisted. To properly transfer the desired torque to the user, the structure of the robot has to deal with unpredictable variations of human-joint position and orientation, introduced by the variable link lengths and other factors. A macroscopic regulation of the robot link lengths [19], [38], [39] can only alleviate the problem, because an accurate measure of human link lengths cannot be obtained “in vivo” without complex instrumental analysis. Such regulations introduce unknown errors, which depend on the skills of the experimenter in evaluating the user anthropometry and fitting the user inside the exoskeleton. In addition, regulations of robot links cannot compensate for all the variable misalignments. Finally, adjustable link lengths can affect the mechanical solidity of the robot and vary its dynamical parameters (e.g. centre of gravity positions and inertia tensors).

Part of these problems have been solved in literature by introducing additional passive DOFs [22], [25], [26], [33], [34]. These exoskeletons can withstand large axes misalignments and can be worn correctly by people with different body sizes with no need of manual adjustments. If an exoskeleton includes sufficient passive DOFs to cover a wide variability in limb sizes, manual regulations become useless, and can be excluded from the design. In practical cases, including some degree of manual regulation of link lengths in a SAM can result in a relaxation of the maximum deviations δ the designer wants to cover with the passive DOFs. Depending on the specific joint and bulk of the robot, this hybrid solution could be beneficial.

Very often, passive DOFs have been used as “patches” of the main active DOFs, with the design being driven on case-specific considerations. In [26] authors proposed dimensional constraints over active and passive DOFs in an exoskeleton chain (similar to those expressed by (3)). In this work, the main focus is about the “hyperstaticity” of the coupled human-robot system (i.e. the motion impediment due to an incorrect alignment of homologous robot-human joint axes), which represents a particular case of what the present paper calls “misalignment”. In order to avoid hyperstaticity, authors of [26] propose to endow the attachment points with passive DOFs. The mobility of these DOFs respects dimensional constraints so that each closed kinematic chain joining the human, the robot and a fixed reference frame (as the loop $O_F - O_0 - E - O - O_0 - O_F$ in Fig. 2) has the same mobility of the human chain. The main conceptual difference with our work is that [26] does not stress the possibility of misplacement of the human joints’ axes from the supposed positions. If such deviations occur, and the dimensional constraints on mobility of each chain are respected, they will be absorbed by the passive attachments, permitting human limb motion. In [26], an initial choice over the $q_c$ and $q_a$ partition (equation (7)) is made, inherently given by the anthropomorphic structure of the exoskeleton: $q_c$ are the robotic joints, and are in a one-to-one correspondence to the human joints $q_h$, while the $q_a$ lies in...
This paper presented a complete analytical treatment of the problem of misalignment between a robotic chain and another modelling the human limb to be assisted (i.e., a formal description of the coupled kineto-static problem). This approach can be applied to solve most of the critical issues in the design of a wearable-robot structure: flexibility to variable user anthropometry, evaluation of natural workspace impairment, and capability of transmitting assisting torque to the user joints. We presented the planar decomposition approach for a spatial 2R human joint case, and formally demonstrated that it corresponds to splitting the design of a full 3D SAM in two simpler and lower-grade 2D SAMs. This problem was never treated in past designs of exoskeletal platforms with passive DOFs, but should be considered in the design phase.

B. Conclusions and future works

This paper presented a complete analytical treatment of the problem of misalignment between a robotic chain and another modelling the human limb to be assisted (i.e., a formal description of the coupled kineto-static problem). This approach can be applied to solve most of the critical issues in the design of a wearable-robot structure: flexibility to variable user anthropometry, evaluation of natural workspace impairment, and capability of transmitting assisting torque to the user joints. We presented the planar decomposition approach for a spatial 2R human joint case, and formally demonstrated that it corresponds to splitting the design of a full 3D SAM in two simpler and lower-grade 2D SAMs. Finally, we proposed a possible architecture for an exoskeleton assisting a spatial 2R human joint (see Fig. 7). While this methodology was developed as a tool for wearable-robot design, it could be applied to all coupled serial kinematic chains that need an active assistance support along some
axes while not constraining the others in term of force and displacement.

APPENDIX A

In this Appendix, detailed mathematical derivations of matrix transformations related to the 2R joint example (Section IV) are shown. In order to get (26) in Section IV-A, each of the factor of (3) is calculated. Variables and axes are defined in Fig. 4(a) and 4(b). The homogeneous transformation matrix from the $O_0$ frame to the $E$ frame results from the composition of rotation $\tilde{q}_1$ about $\tilde{z}_0$ axis, $\tilde{q}_2$ about $\tilde{z}_1$ axis and $\varepsilon$ about $\tilde{x}_e$ axis:

$$0^T_0(S, \delta) = \begin{bmatrix} R^0_0 & [l_x(S)] & I_3 \delta_1 \\ 0^T & [l_y(S)] & \delta_2 \\ 0 & [l_z(S)] & \delta_3 \end{bmatrix}. \quad (31)$$

$$\hat{0}T_E(q, \delta) = \hat{0}T_1(\hat{q}_1)\hat{1}T_E(\hat{q}_2, \varepsilon) = \begin{bmatrix} R_\varepsilon(\hat{q}_1)R_\eta(-\frac{\pi}{2}) & 0 \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} R_\varepsilon(\hat{q}_2)R_\eta(\varepsilon) & 0 \\ 0^T & 1 \end{bmatrix} = \begin{bmatrix} R_\varepsilon(\hat{q}_1)R_\eta(-\frac{\pi}{2}) & 0 \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} R_\varepsilon(\hat{q}_2) & 0 \\ 0^T & 1 \end{bmatrix} = \hat{0}T_1(\hat{q}_1)\hat{1}T_E(\hat{q}_2, 0) = \begin{bmatrix} R_\varepsilon(\hat{q}_1) & 0 \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} R_\varepsilon(\hat{q}_2) & 0 \\ 0^T & 1 \end{bmatrix}, \quad (32)$$

$$\hat{e}T_e(S) = \begin{bmatrix} R^e_e & \begin{bmatrix} l_{ht}(S) \end{bmatrix} \\ 0^T & 1 \end{bmatrix}. \quad (33)$$

In order to get (27) in Section IV-B, we start from the central factors of equation (26), and manipulate by means of (32):

$$\begin{bmatrix} I_3 & \delta \\ 0^T & 1 \end{bmatrix} \hat{0}T_E(q, 0) = \begin{bmatrix} I_3 & \delta \\ 0^T & 1 \end{bmatrix} \hat{0}T_1(\hat{q}_1)\hat{1}T_E(\hat{q}_2, 0) = \begin{bmatrix} I_3 & \delta \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} I_3 & \delta \\ 0^T & 1 \end{bmatrix} \hat{1}T_E(\hat{q}_2, 0) = \begin{bmatrix} I_3 & \delta \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} R_\varepsilon(\hat{q}_1) & 0 \\ 0^T & 1 \end{bmatrix} \hat{1}T_E(\hat{q}_2, 0) = \begin{bmatrix} I_3 & \delta \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} \tilde{R}_\varepsilon(\hat{q}_1)R_\eta(\frac{\pi}{2}) & \tilde{R}_\varepsilon(\hat{q}_2) \tilde{\delta} \\ 0^T & 1 \end{bmatrix} \hat{1}T_E(\hat{q}_2, 0) = \begin{bmatrix} I_3 & \delta \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} I_3 & \delta \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} \tilde{R}_\varepsilon(\hat{q}_1)R_\eta(\frac{\pi}{2}) & \tilde{R}_\varepsilon(\hat{q}_2) \tilde{\delta} \\ 0^T & 1 \end{bmatrix} \hat{1}T_E(\hat{q}_2, 0). \quad (34)$$

The vector $R_\varepsilon(\frac{\pi}{2})R_\eta(-\hat{q}_1)\delta$ is the human joint centre displacement represented along the frame $(O_\tilde{x}_0 - \tilde{x}_1, \tilde{z}_1, \tilde{z}_1)$, and we will indicate its components with the symbols $\delta^*_\varepsilon$; for our purposes, we detach the $\pi_1$ co-planar components $\delta^*_1$ and $\delta^*_2$ from $\delta^*_\varepsilon$:

$$\begin{bmatrix} \tilde{R}_\varepsilon(\hat{q}_1) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \tilde{S}_1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \tilde{R}_\varepsilon(\hat{q}_2) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \tilde{S}_1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The product $R_\varepsilon(\hat{q}_1)R_\eta\left(-\frac{\pi}{2}\right) \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ gives

$$R_\varepsilon(\hat{q}_1)R_\eta\left(-\frac{\pi}{2}\right) \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \tilde{R}_\varepsilon(\hat{q}_1) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \delta^*_\varepsilon = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \tilde{R}_\varepsilon(\hat{q}_1) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \delta^*_\varepsilon$$

The important feature is the absence of the third component, meaning that we decoupled the 3D problem into two 2D subproblems. Substituting (36) into (35) leads to (27).

APPENDIX B

In this Appendix, calculations related to the 1R joint example (Section V) are given. Equation (28) in section V-A comes from solution of equation (3); for both left and right terms, Denavit-Hartenberg parameter are needed (see Fig. 5: $z_0$ lays on the plane, whereas $z_1$, $z_2$, $z_3$, $\tilde{z}_0$ and $\tilde{z}_e$ are normal to the plane and pass through $R_1$, $R_2$, $E$, $R_3$ and $E$ respectively, pointing inward the drawing). They are given in table I.

<table>
<thead>
<tr>
<th>$\text{Joint}$</th>
<th>$\alpha$</th>
<th>$\alpha'$</th>
<th>$d$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$0$</td>
<td>$\frac{\pi}{2}$</td>
<td>$q_1 + \alpha$</td>
<td>$\frac{\pi}{2}$</td>
</tr>
<tr>
<td>2</td>
<td>$l_b$</td>
<td>$0$</td>
<td>$0$</td>
<td>$q_2 + \frac{\pi}{2}$</td>
</tr>
<tr>
<td>3</td>
<td>$l_c$</td>
<td>$0$</td>
<td>$0$</td>
<td>$q_3$</td>
</tr>
<tr>
<td>1</td>
<td>$0$</td>
<td>$0$</td>
<td>$\theta$</td>
<td>$\theta$</td>
</tr>
</tbody>
</table>

TABLE I

DENAVIT-HARTENBERG PARAMETERS FOR CHAINS IN FIG. 5.
Substituting (37), (38), (39), and (40), into (3) and solving for $T_f$ in (27).

$0T_f(q_1, q_2, q_3) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & q_1 + l_a & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -s_2 - c_2 0 l_b s_2 \\ c_2 - s_2 0 l_b c_2 \\ c_3 - s_3 0 l_c c_3 \\ s_3 c_3 0 l_s s_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -s_23 - c_23 0 & -l_b s_2 - l_c s_23 \\ c_23 - s_23 0 & q_1 + l_a + l_b c_2 + l_c c_23 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (37)$

whereas the factors of $M(S, \delta, \mathbf{q})$ are:

$0T_f(\mathbf{q}, \mathbf{d}) = \begin{bmatrix} \dot{c} & -\dot{s} & 0 & 0 \\ \dot{s} & \dot{c} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, \quad (38)

$T_f(S) = \begin{bmatrix} 0 & 1 & 0 & l_b \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, \quad (39)

$0T_f(S, \delta) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -l - \delta_2 \\ 1 & 0 & 0 & l_0 + \delta_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, \quad (40)

Substituting (37), (38), (39), and (40), into (3) and solving gives the expression for $f$ in (27).

REFERENCES


